AN EXTREMAL PROBLEM IN THE THEORY OF HARDY FUNCTIONS

BY

H. DYM

ABSTRACT

It is proved that if j is an inner function and $\rho(T) = \sup |\int (e^{i\gamma T}/j(\gamma))f(\gamma)d\gamma|$ over f in the unit ball of H^1, then either $\rho(T) \equiv 1$ for all $T \geqq 0$, or else $\rho(T) \downarrow 0$ exponentially fast as $T \uparrow \infty$. The inner functions j corresponding to each alternative are classified.

1. Introduction

The purpose of this paper is to study the limit, as $T \uparrow \infty$, of

$$\rho(T) = \sup \left| \int e^{i\gamma T} \frac{f(\gamma)}{j(\gamma)} d\gamma \right| : f \in H^1, \|f\| = 1$$

$$= \inf \left\| \frac{e^{i\gamma T}}{j(\gamma)} - k(\gamma) \right\|_\infty : k \in H^\infty,$$

in which the limits of integration (in this and all other such unmarked integrals) are $\pm\infty$, H^p designates the space of Hardy functions over the upper half-plane, and j is a so-called *inner* function. The function $\rho(T)$ arises in the study of the asymptotic orthogonality of (a class of) weakly stationary processes; it measures the cosine of the angle between the past $[x(t): t \leqq 0]$ and the future $[x(t): t \geqq T]$. (See Ibragimov [6], Helson-Sarason [4] and Dym-McKean [3] for additional information.) Besides this, $\rho(T)$ enters in the study of the compactness of translation restricted to certain subspaces of $L^2(0, \infty)$ (See Koosis [7] and Lax [8]).

In this paper the following two theorems are proved:

Received July 5, 1974

THEOREM 1.

$$\rho(T+S) \leqq \rho(T)\rho(S)$$

for every choice of $S \geqq 0$ *and* $T \geqq 0$, *and so either* $\rho \equiv 1$ *or else* $\rho(T) \to 0$ *exponentially fast, as* $T \uparrow \infty$.

THEOREM 2. $\rho(T) \to 0$ *as* $T \to \infty$ *if and only if the inner function* j *has no singular factors and its roots* $\omega_n = a_n + ib_n$ *meet the condition*

$$\sum_{n=1}^{\infty} \frac{b_n}{|\gamma - \omega_n|^2} \leqq M < \infty$$

for all real γ.

2. Proof of Theorem 1

The proof is broken into a number of steps. In these H^2 is to be thought of as a closed subspace of the Hilbert space $L^2(R^1, d\gamma)$; $(\,,\,)$ denotes the standard inner product and $\omega^* = a - ib$ the complex conjugate of $\omega = a + ib$. Hoffman [5] and Duren [2] are the standard references for H^p spaces; the first two chapters of Dym-McKean [3] contain a succinct treatment of everything along these lines which is needed for this paper.

STEP 1. $f \in H^2 \ominus jH^2$ if and only if $(f/j)^* \in H^2 \ominus jH^2$.

PROOF. If $f \in H^2 \ominus jH^2$, then the map

$$V: f \to (f/j)^*$$

sends $H^2 \ominus jH^2$ into itself. The statement drops out from the fact that V^2 is the identity.

STEP 2. Let $\mathfrak{P}$ denote the orthogonal projection of $L^2(R^1)$ onto H^2 and let

$$\mathfrak{P}_T = \mathfrak{P} e^{-i\gamma T}, \text{ restricted to } H^2 \ominus jH^2.$$

Then

$$\begin{aligned} \rho(T) &= \|\mathfrak{P}_T\| \\ &= \sup \|\mathfrak{P} e^{-i\gamma T} f\| : f \in H^2 \ominus jH^2 \quad \text{and} \quad \|f\| = 1. \end{aligned}$$

PROOF. Let $A\,[B]$ denote the unit sphere of $H^2 \ominus jH^2 [H^2]$. Then, by step 1,

$$\sup_{f\in A}\|\mathfrak{P}_T f\| = \sup_{f\in A}\|\mathfrak{P}_T Vf\|$$

$$= \sup_{f\in A}\sup_{g\in B}|(g,\mathfrak{P}_T Vf)|$$

$$= \sup_{f\in A}\sup_{g\in B}\left|\int \frac{e^{i\gamma T}}{j(\gamma)} f(\gamma)g(\gamma)d\gamma\right|$$

$$= \sup_{f\in B}\sup_{g\in B}\left|\int \frac{e^{i\gamma T}}{j(\gamma)} f(\gamma)g(\gamma)d\gamma\right|$$

$$= \rho(T).$$

The change of function class from $f \in A$ to $f \in B$ in the next to last line does not affect the equality because

$$\int \frac{e^{i\gamma T}}{j(\gamma)} f_1(\gamma)g(\gamma)d\gamma = 0$$

for $f_1 \in jH^2$. The final identification with $\rho(T)$ follows from the fact that the product of two H^2 functions belongs to H^1 and that conversely every $h \in H^1$ can be expressed as the product h_1h_2 of two H^2 functions with

$$\|h_1h_2\|_1 = \|h_1\|_2\,\|h_2\|_2.$$

STEP 3. If $T \geqq 0$ and $S \geqq 0$, then $\mathfrak{P}_T$ maps $H^2 \ominus jH^2$ into itself, and $\mathfrak{P}_T\mathfrak{P}_S = \mathfrak{P}_{T+S}$.†

PROOF. If $f \in H^2 \ominus jH^2$ and $g \in H^2$, then

$$(\mathfrak{P}_T f, jg) = (e^{-i\gamma T}f, jg)$$

$$= (f, e^{i\gamma T}jg) = 0$$

for $T \geqq 0$. This proves that $\mathfrak{P}_T f$ is orthogonal to jH^2 and so too that $\mathfrak{P}_T$ maps $H^2 \ominus jH^2$ into itself. Therefore $\mathfrak{P}_S\mathfrak{P}_T$ is well defined since the range of $\mathfrak{P}_T$ is included in the domain of $\mathfrak{P}_S$. The semigroup identity is now easily verified: for every $g \in H^2$ and $f \in H^2 \ominus jH^2$,

$$(g, \mathfrak{P}_S\mathfrak{P}_T f) = (e^{i\gamma S}g, \mathfrak{P}_T f)$$

$$= (e^{i\gamma S}g, e^{-i\gamma T}f)$$

$$= (g, e^{-i\gamma(S+T)}f)$$

$$= (g, \mathfrak{P}_{S+T}f).$$

† This is perhaps more transparent if you bear in mind that $\mathfrak{P}_T$ is unitarily equivalent to left translation restricted to a left invariant subspace of $L^2[0,\infty)$. A direct proof is furnished in order to both simplify the exposition and to help set the notation.

STEP 4. Step 4 is to combine the implications of steps 2 and 3 to prove the Theorem:

$$\begin{aligned}\rho(S+T) &= \|\mathfrak{P}_{S+T}\| \\ &= \|\mathfrak{P}_S\mathfrak{P}_T\| \\ &\leqq \|\mathfrak{P}_S\|\,\|\mathfrak{P}_T\| = \rho(S)\rho(T).\end{aligned}$$

3. Proof of Theorem 2.

Let $\mathfrak{H}_\infty$ denote the closure in $L^\infty(R^1)$ of $\cup_{T\geqq 0} e^{-i\gamma T}H^\infty$ and observe that $\rho(T)\to 0$ as $T\uparrow\infty$ if and only if $1/j$ belongs to $\mathfrak{H}_\infty$.

LEMMA 1. (Mentioned in [1]) *$\mathfrak{H}_\infty$ is a subalgebra of L^∞ which contains the class of functions f which are bounded and uniformly continuous on R^1.*

PROOF. If f is bounded and uniformly continuous on R^1, then

$$f_T(\omega) = \frac{e^{i\omega T}}{\pi}\int f(\gamma)2\frac{\sin^2\frac{T}{2}(\omega-\gamma)}{T(\omega-\gamma)^2}d\gamma$$

belongs to H^∞ for every $T>0$ and

$$\|e^{-i\omega T}f_T - f\|_\infty \to 0,$$

as $T\uparrow\infty$. This proves the second assertion; the first is easily verified.

LEMMA 2. (Adapted from [7]) *If j is an inner function and $1/j$ belongs to $\mathfrak{H}_\infty$, then*

$$\left|\frac{1}{j(a+ib)}\right| \leqq L < \infty$$

in a horizontal strip $0<b\leqq\delta$.

PROOF. By hypothesis, you can find a function $k\in H^\infty$ and a number $T>0$ such that

$$\left|\frac{1}{j(\gamma)} - e^{-i\gamma T}k(\gamma)\right| \leqq \epsilon < \frac{1}{2} \quad (\gamma\in R^1)$$

for any such preassigned $\epsilon>0$. Now, as j and $e^{-i\gamma T}$ have modulus 1 on R^1, you see that

$$|e^{i\gamma T} - j(\gamma)k(\gamma)| \leqq \epsilon \qquad (\gamma\in R^1)$$

and

$$|k(\gamma)| \leqq 1+\epsilon \qquad (\gamma \in R^1).$$

Moreover, these last two bounds propagate to all of R^{2+} since the functions being bounded from above belong to H^∞. In particular,

$$|e^{i(a+ib)T} - j(a+ib)k(a+ib)| \leqq \epsilon$$

for all $b \geqq 0$, and so

$$|1 - e^{-i(a+ib)T} j(a+ib)k(a+ib)| \leqq \epsilon e^{bT} < \tfrac{1}{2}$$

for sufficiently small $b > 0$. But this means that

$$|e^{-i(a+ib)T} j(a+ib)k(a+ib)| \geqq \tfrac{1}{2}$$

for such b, or what is the same

$$\left|\frac{1}{j(a+ib)}\right| \leqq 2\, e^{bT}\, |k(a+ib)|$$

$$\leqq \frac{1+\epsilon}{\epsilon}\,.$$

LEMMA 3.† *If $\zeta = u + iv \in R^{2+}$, if $\xi \in \overline{R^{2+}}$, and if $|\xi/\zeta| \leqq \frac{1}{2}$, then there exists a complex number α with $|\alpha| \leqq 2$, such that*

$$\frac{1-\xi/\zeta^*}{1-\xi/\zeta} = \exp\left\{\frac{-2iv\xi}{|\zeta|^2}\left[1+\alpha\frac{|\xi|}{|\zeta|}\right]\right\}\,.$$

PROOF. To begin with

$$\frac{1-\xi/\zeta^*}{1-\xi/\zeta} = \exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{\xi}{\zeta}\right)^n - \left(\frac{\xi}{\zeta^*}\right)^n\right]\right\}$$

$$= \exp\left\{\frac{-2iv\xi}{|\zeta|^2} + \sum_{n=2}^{\infty}\frac{\xi^n}{n}\,\frac{(\zeta^*)^n - \zeta^n}{|\zeta|^{2n}}\right\}\,.$$

The next step is to estimate the second term in the exponential with the help of the inequality

$$|\zeta^n - (\zeta^*)^n| = \left|\int_{-1}^{1} n(u+isv)^{n-1} iv\,ds\right| \leqq n\,|\zeta|^{n-1} 2v.$$

This gives you the bound

† $R^{2+}[\overline{R^{2+}}]$ stands for the open (closed) upper-half plane.

$$\left|\sum_{n=2}^{\infty} \frac{\xi^n}{n} \frac{(\zeta^*)^n - \zeta^n}{|\zeta|^{2n}}\right| \leqq \frac{2v|\xi|}{|\zeta|^2} \sum_{n=2}^{\infty} |\xi/\zeta|^{n-1}$$

$$= \frac{2v|\xi|}{|\zeta|^2} \frac{|\xi/\zeta|}{1-|\xi/\zeta|}$$

$$\leqq \frac{2v|\xi|}{|\zeta|^2} 2|\xi/\zeta|.$$

The final statement follows upon combining estimates.

LEMMA 4. *If $j = P$ is a Blaschke product with roots $\omega_n = a_n + ib_n \in R^{2+}$, then $1/j$ belongs to $\mathfrak{H}_\infty$ if and only if*

$$\sum \frac{b_n}{|\gamma - \omega_n|^2} \leqq M < \infty$$

for all $\gamma \in R^1$.

PROOF. If $1/P$ belongs to $\mathfrak{H}_\infty$, then by Lemma 2,

$$\left|\frac{1}{P(a+ib)}\right| \leqq L < \infty$$

in a horizontal strip $0 \leqq b \leqq \delta$ of width $\delta > 0$. In particular, this means that the roots ω_n of P lie above this strip: $b_n \geqq \delta$, and so

$$\frac{b}{|\gamma - \omega_n|} \leqq \frac{b}{b_n} \leqq \frac{b}{\delta} \leqq \frac{1}{2}$$

for $\gamma \in R^1$ and $0 \leqq b \leqq \delta/2$. Therefore (as 0 is not a limit point of the roots ω_n) the product P can be written in the form

$$P(\omega) = \prod_{n=1}^{\infty} \frac{1 - \omega/\omega_n}{1 - \omega/\omega_n^*}$$

and Lemma 3 provides a lower bound for each term in the partial product

$$1/P_N(\omega) = \prod_{n=1}^{N} \frac{1 - \omega/\omega_n^*}{1 - \omega/\omega_n}$$

in the smaller strip $\omega = a + ib$ with $0 \leqq b \leqq \delta/4$:

$$L \geqq 1/|P_N(a+ib)|$$

$$= \left| \prod_{n=1}^{N} \frac{1 - \dfrac{ib}{\omega_n^* - a}}{1 - \dfrac{ib}{\omega_n - a}} \right|$$

$$= \left| \exp\left\{ \sum_{n=1}^{N} \frac{2b_n b}{|\omega_n - a|^2} \left[1 + \frac{\alpha_n b}{|\omega_n - a|} \right] \right\} \right|$$

$$\geqq \exp\left\{ \sum_{n=1}^{N} \frac{b_n b}{|\omega_n - a|^2} \right\} .$$

To finish, let $N \uparrow \infty$.

It remains to show that $1/P$ is uniformly continuous on R^1 under the stated root conditions. But, for real γ and real ϵ,

$$\left| \frac{1}{P(\gamma + \epsilon)} - \frac{1}{P(\gamma)} \right| = \left| \frac{P(\gamma)}{P(\gamma + \epsilon)} - 1 \right|$$

$$= \left| \prod_{n=1}^{\infty} \frac{1 - \dfrac{\epsilon}{\omega_n^* - \gamma}}{1 - \dfrac{\epsilon}{\omega_n - \gamma}} - 1 \right|$$

$$= |e^{i\phi} - 1|$$

$$\leqq |\phi|$$

where

$$i\phi = \sum_{n=1}^{\infty} \frac{-2ib_n\epsilon}{|\omega_n - \gamma|^2} \left[1 + \frac{\alpha_n \epsilon}{|\omega_n - \gamma|} \right]$$

by Lemma 3, assuming as you may, that $|\epsilon/(\omega_n - \gamma)| \leqq \frac{1}{2}$. The bound

$$|i\phi| \leqq 4\epsilon \sum \frac{b_n}{|\omega_n - \gamma|^2} \leqq 4\epsilon M$$

is now self-evident as is the uniform continuity of $1/P$:

$$\left| \frac{1}{P(\gamma + \epsilon)} - \frac{1}{P(\gamma)} \right| \leqq 4\epsilon M.$$

LEMMA 5. *If $j = S$ is purely singular, then $1/j$ does not belong to $\mathfrak{H}_\infty$.*

PROOF. In the upper-half plane, a singular inner function S can be expressed in the form

$$S(\omega) = \exp\left\{ -\frac{1}{\pi i} \int \left[\frac{1}{\gamma - \omega} - \frac{\gamma}{\gamma^2 + 1} \right] dF(\gamma) \right\}$$

in which F is a non-decreasing singular function which is subject to the constraint $\int (\gamma^2 + 1)^{-1} dF(\gamma) < \infty$. Now if $1/S$ belongs to $\mathfrak{H}_\infty$, then by Lemma 2, the harmonic function

$$\begin{aligned} u(a,b) &= -\log|S(a+ib)| \\ &= \frac{b}{\pi} \int \frac{1}{(\gamma - a)^2 + b^2} \, dF(\gamma) \end{aligned}$$

is uniformly bounded in the horizontal strip $0 < b \leqq \delta$. This clearly rules out jumps in F and at the same time permits you to invoke the dominated convergence theorem in order to evaluate

$$\begin{aligned} F(\gamma_2) - F(\gamma_1) &= \lim_{b \downarrow 0} \int_{\gamma_1}^{\gamma_2} u(a,b) da \\ &= \int_{\gamma_1}^{\gamma_2} \lim_{b \downarrow 0} u(a,b) da \\ &= \int_{\gamma_1}^{\gamma_2} F'(a) da = 0. \end{aligned}$$

COROLLARY. *If $1/j \in \mathfrak{H}_\infty$, then the inner function j has no singular factors.*

PROOF. If $j = Sj_1$ is the product of a singular inner function S with a second inner function j_1 and if $1/j$ belongs to $\mathfrak{H}_\infty$, then so does

$$\frac{1}{S} = j_1 \cdot \frac{1}{j}$$

in contradiction to Lemma 5.

The proof of the theorem is now easily completed with the help of Lemmas $1 \cdots 5$ and the last corollary. The latter implies that if $1/j \in \mathfrak{H}_\infty$, then j is a constant multiple of $Pe^{i\gamma c}$ with P a Blaschke product and $c \geqq 0$ a constant. Therefore, $1/P$ also belongs to $\mathfrak{H}_\infty$, and so the bound on the roots follows by Lemma 4. Conversely, if the root constraint is met, then $1/P$ belongs to $\mathfrak{H}_\infty$ as does $1/j$ for any inner function j with Blaschke part P and no singular factors.

REFERENCES

1. A. Devinatz, *On Wiener-Hopf operators*, Proceedings Irvine Conference on Functional Analysis, Thompson, Washington, D.C., 1967 , pp. 81–118.